Discrete Optimization
ISyE 6662 - Spring 2023
Homework 4

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1. Let (N, E) be an undirected network, and consider the polytope

$$P_d = \left\{ x \in \mathbb{R}_+^E \ : \ \sum_{e \in \delta(i)} x_e \le d, \ i \in N \right\},$$

for $d \in \mathbb{N}$

a) Suppose the network is bipartite; show that P_1 is integral by showing that any fractional point cannot be extreme. Hint: Start by assuming the network has a cycle. Then argue the acyclic case.

Answer: Let $x \in P_1$ be a fractional solution. Let $E(x) = \{e \in E : x_e \in (0,1)\}$ be the set of edges associated to a fraction coordinate of x. Let $\varepsilon = \min_{e \in E(x)} \big(\min\{x_e, 1 - x_e\}\big)$. Recall that a bipartite graph cannot contain odd cycles. Then, there are two possibilities for the graph $G(x) := (N, E(x))$:

i. $G(x)$ has an even cycle C. Let $i_1 i_2 \cdots i_{2k} i_1$ be the of nodes C. We can define two other solutions $x^1, x^2 \in P_1$ such that $x = \frac{x^1 + x^2}{2}$:

$$x_e^1 = \begin{cases} x_e + \epsilon, & \text{if } e = (i_r, i_{r+1}) \text{ and } r \text{ is even,} \\ x_e - \epsilon, & \text{if } e = (i_r, i_{r+1}) \text{ and } r \text{ is odd or } e = (i_{2k}, i_1), \\ x_e, & \text{if } e \text{ does not belong to } C, \end{cases}$$
$$x_e^2 = \begin{cases} x_e - \epsilon, & \text{if } e = (i_r, i_{r+1}) \text{ and } r \text{ is even,} \\ x_e + \epsilon, & \text{if } e = (i_r, i_{r+1}) \text{ and } r \text{ is odd or } e = (i_{2k}, i_1), \\ x_e, & \text{if } e \text{ does not belong to } C. \end{cases} \tag{1}$$

Hence, x cannot be an extreme point.

ii. $G(x)$ is acyclical. Let $P = i_1 \cdots i_n$ be a maximal path in $G(x)$. Note that x_e must be 0 for any edge in $\delta(i_1) \cup \delta(i_n) \backslash E(x)$. Indeed, if x_e is 1 for some edge $e \in \delta(i_1) \backslash E(x)$ then the constraint $\sum_{e \in \delta(i_1)} x_e \le 1$ implies that the solution $x_{e'}$ for the edge $e' = (i_1, i_2)$ cannot be fraction. Similarly, for $e \in \delta(i_n) \backslash E(x)$. Thus, x_e must be 0 for any edge in $\delta(i_1) \cup \delta(i_n) \backslash E(x)$. We define two other solutions $x^1, x^2 \in P_1$ such that $x = \frac{x^1 + x^2}{2}$ analogously to (1):

$$x_e^1 = \begin{cases} x_e + \epsilon, & \text{if } e = (i_r, i_{r+1}) \text{ and } r \text{ is even,} \\ x_e - \epsilon, & \text{if } e = (i_r, i_{r+1}) \text{ and } r \text{ is odd,} \\ x_e, & \text{if } e \text{ does not belong to } P, \end{cases}$$
$$x_e^2 = \begin{cases} x_e - \epsilon, & \text{if } e = (i_r, i_{r+1}) \text{ and } r \text{ is even,} \\ x_e + \epsilon, & \text{if } e = (i_r, i_{r+1}) \text{ and } r \text{ is odd,} \\ x_e, & \text{if } e \text{ does not belong to } P. \end{cases} \tag{2}$$

Hence, x cannot be an extreme point.

b) Now suppose the network is not necessarily bipartite; prove that P_2 is integral, by again arguing that a fractional point cannot be extreme. Hint: Start with your argument from (a) and then think of how many variables can have positive value.

Answer: Given a graph $G = (N, E)$, we create a bipartite graph $\overline{G} = (N' \cup N'', \overline{E})$ with partitions N' and N'' by duplicating each node $i \in N$ into i' and i'' in N' and N'', respectively. For each edge $\{i, j\} \in E$ we create the edges $\{i', j''\}$ and $\{i'', j'\}$ in the sets E_1 and E_2, respectively. We define the set of edges $\overline{E}$ as $E_1 \cup E_2$. Consider the following polytope Q:

$$Q = \left\{ y \in \mathbb{R}_+^{\overline{E}} \; : \sum_{\overline{e} \in \delta_{\overline{G}}(k)} y_{\overline{e}} \le 1, \;\forall k \in N' \cup N'' \right\}. \tag{3}$$

From question (a), we know that Q is an integral polytope. Let $A : \mathbb{R}^{\overline{E}} \to \mathbb{R}^E$ be the following linear transformation:

$$[A(y)]_e := y_{e_1} + y_{e_2}, \tag{4}$$

where $e = \{i, j\} \in E$, $e_1 = \{i', j''\} \in E_1$, and $e_2 = \{i'', j'\} \in E_2$. It is enough to prove that $P_2 = A(Q)$ since the extreme points of the image of a polyhedron is the image of the extreme points, and the sum of two integral vectors is an integral vector.

Indeed, let $x \in P_2$. Then, define $y \in \mathbb{R}_+^{\overline{E}}$ as follows:

$$y_{\overline{e}} = \begin{cases} x_e/2, & \text{if } \overline{e} = \{i', j''\} \in E_1, \text{ where } e = \{i, j\}, \\ x_e/2, & \text{if } \overline{e} = \{i'', j'\} \in E_2, \text{ where } e = \{i, j\}. \end{cases}$$

It follows from the definition of the bipartite graph $\overline{G}$ that $\sum_{\overline{e} \in \delta_{\overline{G}}(k)} y_{\overline{e}} \le 1$. So, $P_2 \subseteq A(Q)$. Conversely, for any $y \in Q$, we have that x defined as $A(y)$ is such that

$$\sum_{e \in \delta_G(i)} x_e = \sum_{e \in \delta_G(i)} (y_{e_1} + y_{e_2}) = \sum_{\overline{e} \in \delta_{\overline{G}}(i')} y_{\overline{e}} + \sum_{\overline{e} \in \delta_{\overline{G}}(i'')} y_{\overline{e}} \le 2.$$

Thus, $P_2 \supseteq A(Q)$. Hence, we conclude that $P_2 = A(Q)$.

2. Consider polytopes $P_k = \{x \in \mathbb{R}^n : A^k x \le b^k\}$ for $k = 1, \ldots, K$, and recall the copies method: We model $\bigcup_k P_k$ with

$$Q_I = \left\{ x, x^1, \ldots, x^K \in \mathbb{R}^n; \; z \in \{0,1\}^K : x = \sum_{k=1}^K x^k; \; \sum_{k=1}^K z_k = 1; \; A^k x^k \le b^k z_k, \; \forall k \right\}$$

Let Q be the linear relaxation of Q_I, where $z \in [0,1]^K$. Prove that $Q = \text{conv}(Q_I)$, and thus $\text{proj}_x(Q) = \text{conv}(\bigcup_k P_k)$ Hint: If you can prove it for $K = 2$ you can prove it for any K.

Answer: We observe that $\text{conv}(Q_I) \subseteq Q$, since Q is the linear relaxation of Q_I. To prove the reverse inclusion, we need to show that any vector of Q is the convex combination of vectors in Q_I. Let $(x, x^1, \ldots, x^k, z) \in Q$ and let $I(z) = \big\{i \in \{1, \ldots, k\} : z_i > 0\big\}$. Note that

$$\begin{aligned} & I(z) \text{ is nonempty}, \\ & \widetilde{x}^i := \frac{1}{z_i} x^i \in P_i, \quad \forall i \in I(z), \quad \text{and} \\ & x^i = 0, \quad \forall i \in \{1, \ldots, k\} \backslash I(z). \end{aligned}$$

Recall that $(\widetilde{x}, \widetilde{x}^1, \ldots, \widetilde{x}^k, \widetilde{z})$ belongs to Q_I if, and only if,

$$\widetilde{x} = \widetilde{x}^i, \qquad \widetilde{x}^i \in P_i, \qquad \widetilde{x}^j = 0 \;\text{ for all } j \ne i, \qquad \text{and} \qquad \widetilde{z} = e^i \;\text{ for some } i \in \{1, \ldots, k\}, \tag{5}$$

This concludes that $(x, x^1, \ldots, x^k, z)$ is the convex combination of vectors in Q_I, as indicated by (5), with weights $\{z_i\}_{i\in I(z)}$.

For the last part, note that $\operatorname{proj}_x Q_I = \bigcup_k P_k$ and the equality

$$\operatorname{proj}_x \big(\operatorname{conv} A\big) = \operatorname{conv} \big(\operatorname{proj}_x A\big)$$

holds for any subset A. Hence, $\operatorname{proj}_x(Q) = \operatorname{proj}_x(\operatorname{conv} Q_I) = \operatorname{conv}(\bigcup_k P_k)$.

3. For an undirected network (N, E) without isolated nodes, consider the polytope $P \subseteq \mathbb{R}_+$ defined by non-negativity and the constraints $x_i + x_j \le 2$, for $\{i, j\} \in E$.

   a) Suppose the network has a cycle. Show that the constraints defined by the edges of the cycle are linearly independent if and only if the cycle is odd.

   Answer: Let C be the cycle defined by the nodes $i_1 i_2 \cdots i_n$. The coefficient matrix A_n that represents the constraints $x_i + x_j \le 2$ for edges $\{i, j\} \in E(C)$ can be represented as:

$$A_n = \begin{array}{c} \begin{matrix} i_1 & i_2 & i_3 & i_4 & \cdots & i_{n-1} & i_n \end{matrix} \\ \begin{pmatrix} 1 & 1 & & & & & \\ & 1 & 1 & & & & \\ & & 1 & 1 & & & \\ & & & & \ddots & & \\ & & & & & 1 & 1 \\ 1 & & & & & & 1 \end{pmatrix} \end{array} \begin{matrix} \{i_1, i_2\} \\ \{i_2, i_3\} \\ \{i_3, i_4\} \\ \vdots \\ \{i_{n-1}, i_n\} \\ \{i_n, i_1\} \end{matrix} \tag{6}$$

We show by induction that $\det A_n = 2$ if n is odd, and $\det A_n = 0$ if n is even. Indeed, the base cases $\det A_3$ and $\det A_4$ can be easily computed:

$$\det A_3 = \begin{vmatrix} 1 & 1 & \\ & 1 & 1 \\ 1 & & 1 \end{vmatrix} \overset{(L_2 - L_3)}{=} \begin{vmatrix} 1 & 1 & \\ -1 & 1 & 0 \\ 1 & & 1 \end{vmatrix} \overset{(L_1 - L_2)}{=} \begin{vmatrix} 2 & 0 & \\ -1 & 1 & 0 \\ 1 & & 1 \end{vmatrix} = 2,$$

$$\det A_4 = \begin{vmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \\ 1 & & & 1 \end{vmatrix} \overset{(L_3 - L_4)}{=} \begin{vmatrix} 1 & 1 & & \\ & 1 & 1 & \\ -1 & & 1 & 0 \\ 1 & & & 1 \end{vmatrix} \overset{(L_2 - L_3)}{=} \begin{vmatrix} 1 & 1 & & \\ 1 & 1 & 0 & \\ -1 & & 1 & 0 \\ 1 & & & 1 \end{vmatrix} \overset{(L_1 - L_2)}{=} \begin{vmatrix} 0 & 0 & & \\ 1 & 1 & 0 & \\ -1 & & 1 & 0 \\ 1 & & & 1 \end{vmatrix} = 0.$$

We complete our proof by showing that $\det A_n = \det A_{n-2}$:

$$\begin{aligned} \det A_n \quad &= \begin{vmatrix} 1 & 1 & & & & \\ & 1 & 1 & & & \\ & & 1 & 1 & & \\ & & & \ddots & & \\ & & & & 1 & 1 \\ 1 & & & & & 1 \end{vmatrix} \overset{(C_2 - C_1)}{=} \begin{vmatrix} 1 & 0 & & & & \\ & 1 & 1 & & & \\ & & 1 & 1 & & \\ & & & \ddots & & \\ & & & & 1 & 1 \\ 1 & -1 & & & & 1 \end{vmatrix} \\ &\overset{(C_3 - C_2)}{=} \begin{vmatrix} 1 & 0 & & & & \\ & 1 & 0 & & & \\ & & 1 & 1 & & \\ & & & \ddots & & \\ & & & & 1 & 1 \\ 1 & -1 & 1 & & & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & & \\ & & \ddots & \\ & & 1 & 1 \\ 1 & & & 1 \end{vmatrix} = \det A_{n-2}. \end{aligned}$$

b) Use your answer in (a) to show that the polytope is integral, by arguing directly that every extreme point must be integral.

Answer: Let A be the incidence matrix of the constraints $x_i + x_j \leq 2$, for $\{i, j\} \in E$, where the rows and columns represent the edges and nodes of the graph $G = (N, E)$, respectively, as illustrated in Eq. (6). An extreme point x to the polytope $P = \{x \in \mathbb{R}^E : Ax \leq 2 \cdot \mathbb{1},\ x \geq 0\}$ is the unique solution to

$$\begin{aligned} H x_H &= 2 \cdot \mathbb{1}_H, \\ x_{H^\complement} &= 0, \end{aligned}$$

where $H \in \mathbb{R}^{k \times k}$ is a square non-singular submatrix of A, $\mathbb{1}_H \in \mathbb{R}^k$ is a vector of 1's, $x_H \in \mathbb{R}^k$ and $x_{H^\complement} \in \mathbb{R}^{n-k}$ are subvectors of $x \in \mathbb{R}^n$, and $n := |N|$. From the Cramer's rule, we have that

$$(x_H)_i = \frac{\det H_i}{\det H},$$

where H_i is the matrix formed by replacing the i-th column of H by the column vector $2 \cdot \mathbb{1}_H$. Let $\widetilde{H}_i$ be the matrix formed by replacing the i-th column of H by the column vector $\mathbb{1}_H$ (instead of $2 \cdot \mathbb{1}_H$). Note that

$$\det H_i = 2^k \cdot \det \widetilde{H}_i, \quad \text{and} \quad \det \widetilde{H}_i \in \mathbb{Z}.$$

If we prove that $\det H$ is equal to 2^l for some $l \leq k$ then we conclude that x_H is integral.

Below we have some comments regarding $\det H$:

i. By performing row expansions on the determinant of H, we eliminate the rows of H with only one 1. The result is the determinant of an incidence matrix of a subgraph of G.

ii. By performing column expansions on the determinant of H, we eliminate the column of H with only one 1. This result in the determinant of an incidence matrix $\overline{H}$ of a subgraph $G_{\overline{H}}$ for which all the nodes have degree at least 2.

iii. Each connected component of $G_{\overline{H}} = (N_{\overline{H}}, E_{\overline{H}})$ must have the same number of nodes and edges. Indeed, since $|N_{\overline{H}}| = |E_{\overline{H}}|$, if some connected component have more edges than nodes then there is another connected component with more nodes than edges, which implies that there exist at least one node of degree 1. However, this cannot happen because of the previous step. Thus, the incidence matrix $\overline{H}$ is a block diagonal matrix of square matrices $\overline{H}_r \in \mathbb{R}^{k_r \times k_r}$,

$$\det \overline{H} = \begin{bmatrix} \overline{H}_1 & & & \\ & \overline{H}_2 & & \\ & & \ddots & \\ & & & \overline{H}_l \end{bmatrix}, \quad \text{and} \quad \det H = \det \overline{H} = \prod_{r=1}^{l} \det \overline{H}_r, \tag{7}$$

where $l \leq k$ is the number of connected components of $G_{\overline{H}}$ and $\overline{H}_r$ is the incidence matrix of a connected component, for all $1 \leq r \leq l$.

iv. Because the graph $G_{\overline{H}_r} = (N_{\overline{H}_r}, E_{\overline{H}_r})$ is connected, $|N_{\overline{H}_r}| = |E_{\overline{H}_r}|$, and the degree of each node is at least 2 then $G_{\overline{H}_r}$ must be a cycle. In particular, the graph $G_{\overline{H}_r}$ must be an odd cycle, for every $r = 1, \ldots, l$, otherwise the matrix H would be singular.

This concludes our proof that $\det H = 2^l$, for some $l \leq k$.

4. Let A be a TU matrix with full row rank, and let B be a basis of A. Prove that $B^{-1}A$ is TU.

Answer: Recall that a matrix $H \in \mathbb{R}^{m \times n}$ is TU if, and only if, $[H, I_m] \in \mathbb{R}^{m \times (m+n)}$ is *unimodular*. Thus, we show that $C := [B^{-1}A, I_m]$ is unimodular. Let $M \subseteq \{1, \ldots, (n + m)\}$ be a subset of columns of C with cardinality m, that is, $|M| = m$. Let $M_1 := M \cap \{1, \dots, n\}$ and $M_2 := M \cap \{n+1, \dots, m+n\}$. Denote by C_M the matrix formed by the columns in M. Then, we can represent C_M as follows:

$$\begin{aligned} C_M &= \left[(B^{-1}A_j)_{j\in M_1},\ (e_j)_{j\in M_2}\right] \\ &= B^{-1}\left[(A_j)_{j\in M_1}, (B_j)_{j\in M_2}\right] \\ &= B^{-1}\left[A_{M_1}, B_{M_2}\right] \\ &= B^{-1}\left[A_{M_1}, A_{M_2}\right] \end{aligned}$$

where A_j and B_j are the j-th column of A and B, respectively, and e_j is the j-th element of the canonical basis. Since B and $\left[A_{M_1}, A_{M_2}\right]$ are square submatrices of A their determinant are in $\{0, \pm 1\}$. Therefore,

$$\det C_M = \det B^{-1} \cdot \det \left[A_{M_1}, A_{M_2}\right] \in \{0, \pm 1\}.$$

Hence, C is unimodular.

5. Let $A \in \{0,1\}^{n\times(n+1)}$ be a matrix consisting of an identity matrix appended with a column of all 1's. Prove that A is TU directly from the definition, i.e. by showing that all square sub-matrices have determinants in $\{0, \pm 1\}$.

Answer: Let $H \in \mathbb{R}^{k\times k}$ be a square submatrix of A. If H does not contain the last column of A then H is just a submatrix of the identity I_n, so $\det(H) \in \{0, \pm 1\}$. If H contains the last column, one could represent it as $[B,\ \mathbb{1}]$, where $\mathbb{1} \in \mathbb{R}^k$ is a column vector of 1's, and $B \in \mathbb{R}^{k\times(k-1)}$ is a submatrix of I_n. Recall that if we subtract a row from another row it does not change the determinant of a matrix. Thus, we have that

$$\begin{aligned} \det\big([B,\ \mathbb{1}]\big) &= \det\big([\widetilde{B},\ e_1]\big) \\ &= \det \hat{B} \in \{0, \pm 1\}, \end{aligned}$$

where $\widetilde{B}$ is a submatrix of B obtained by subtracting the consecutive rows of B from the first row of B, and $\hat{B} \in \mathbb{R}^{(k-1)\times(k-1)}$ is the submatrix obtained by removing the first row of $\widetilde{B}$.